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Berry's phase, Hannay's angle and coherent states

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Abstract. Through the use of coherent states describing the action-angle variables we establish a strict parallelism between Berry's phase and Hannay's angle. In particular we define a geometrical transport for classical tori which leads to Hannay's angle. As in the case of Berry's phase, the transport can be associated with the minimization of a distance.

1. Introduction

Historically, Berry's phase and Hannay's angle have been introduced in the context of an adiabatic evolution governed by a Hamiltonian $H(\mathbf{X})$ whose parameters \mathbf{X} vary slowly in time. Let us briefly draw a parallel between them in the simple case of one degree of freedom.

In the quantum case, $\{|n, \mathbf{X}\rangle\}$ denotes a continuous (with respect to \mathbf{X}) family of normalized energy eigenstates $|n, \mathbf{X}\rangle$ of the Hamiltonian $H(\mathbf{X})$, with corresponding eigenvalues $E_n(\mathbf{X})$; the choice of a definite vector $|n, \mathbf{X}\rangle$ amounts to the choice of an origin of phase for the vectors which belong to the ray $\overline{|n, \mathbf{X}\rangle}$. It has been shown [1] that the evolved state associated with an initial eigenstate of the Hamiltonian $H(\mathbf{X}(0))$ at time zero is an eigenstate of the Hamiltonian $H(\mathbf{X}(t))$ at time t whose phase

$$\Phi_n(t) = \Phi_n(0) - \hbar^{-1} \int_0^t E_n(X(s)) ds + \gamma_n^B(t) \quad (1)$$

(measured with respect to $\overline{|n, \mathbf{X}(t)\rangle}$) contains two contributions: a dynamical one which is expected and a geometrical one, Berry's phase $\gamma_n^B(t)$, which depends only on the curve Γ which has been followed in the space of parameters between 0 and t . As explained by Simon [2], the geometrical transport which, along Γ , brings an initial eigenstate $|n, \mathbf{X}(0)\rangle$ to the state $\exp(i\gamma_n^B(t))|n, \mathbf{X}(t)\rangle$, can be independently derived from the existence of a 'natural' connection on the line bundle defined over the set of rays $\{\overline{|n, \mathbf{X}\rangle}\}$.

In the classical case, $\{C(I, \mathbf{X})\}$ denotes a continuous family of closed trajectories $C(I, \mathbf{X})$ in the phase space associated with the classical Hamiltonians $H(\mathbf{X})$, and

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$\omega(I, \mathbf{X})$ is the angular velocity on $C(I, \mathbf{X})$; each curve is equipped with a definite origin for the angle variable θ conjugated to the action variable I . It has been shown [3] that, during the adiabatic evolution, a point in phase space follows trajectories of constant action, and that its angular coordinate at time t

$$\theta(t) = \theta(0) + \int_0^t \omega(I, X(s)) ds + \theta_I^H(t) \quad (2)$$

measured on the curve $C(I, \mathbf{X}(t))$, also contains a dynamical contribution and a geometrical one, Hannay's angle $\theta_I^H(t)$. Anandan [4] has interpreted the latter as defining a transport of classical tori in phase space: along Γ , the transported trajectory of $C(I, \mathbf{X}(0))$ is deduced from $C(I, \mathbf{X}(t))$ by an angular shift $\theta_I^H(t)$ of the points of this trajectory. Recently it has been shown that this transport on the phase space bundle over the parameter space is uniquely defined provided the connection flow is Hamiltonian [5].

The relation between Berry's phase and Hannay's angle has been studied from the point of view of semi-classical approximations. By identifying the phase of the quantum wavefunction $\langle q|n, \mathbf{X} \rangle$ with the generating functional of the classical canonical transformation $q, p \rightarrow I, \theta$ (according to the method of Maslov), Berry [6] has established the relation

$$\theta_I^H(t) = -\frac{\partial}{\partial n} \gamma_n^B(t) = -\hbar \frac{\partial}{\partial I} \gamma_n^B(t). \quad (3)$$

The same approximation has been used to argue that the classical counterpart of the quantum geometrical transport is 'the parallel transport of the total surplus area in phase space swept by the system' [7]. An interesting feature of this approach is that it allows \hbar asymptotics for $\theta_I^H(t)$. However, the classical limit does not lead directly to Hannay's angle and its associated transport equation, but to classical quantities which must then be differentiated with respect to the action I . This may lead to difficulties when deriving the classical transport equation.

Our goal in this article is to introduce a formalism which allows a direct derivation of Hannay's angle from Berry's phase and which establishes a direct link between the quantum and the classical transports. Loosely speaking, the philosophy of our approach is to consider classical mechanics as a coarse-grained version of quantum mechanics. The mathematical tool which is well suited for this point of view is that of coherent states [8]. Indeed, one expects that the classical adiabatic evolution and the classical transport in phase space are nothing but the quantum evolution and the quantum transport expressed at the level of these states in the classical limit. We show that this can be established very simply if one chooses those coherent states which are well adapted to the action-angle variables. The interest in introducing such a quantum version of classical mechanics is the possibility of transferring to it the quantum geometrical structures.

The organization of the paper is as follows. In section 2, we define the coherent states $|\alpha, \mathbf{X} \rangle$ associated with an (integrable) Hamiltonian $H(\mathbf{X})$, and show that the complex number α effectively describes the action-angle variables. In section 3, using the states $|\alpha, \mathbf{X} \rangle$, we give a very simple demonstration of (3) starting from the quantum evolution (1) and we show that (3) can be obtained independently from a variational principle. In section 4, we relate the transport of classical trajectories to the quantum transport of coherent states. This transport can be derived either from a geometrical version of the previously quoted variational principle or from the minimization of a distance.

2. Coherent states and action-angle variables

Let $H(\mathbf{X})$ be a (one degree of freedom) quantum Hamiltonian which depends on parameters \mathbf{X} . We suppose that, in the range of variation of these parameters which we consider in the following, the spectrum of $H(\mathbf{X})$ is discrete and non-degenerate. We denote its eigenvectors by $|n, \mathbf{X}\rangle$, the corresponding eigenvalues by $E_n(\mathbf{X})$ and $N(\mathbf{X})$ the excitation number operator is defined by:

$$N(\mathbf{X})|n, \mathbf{X}\rangle = n|n, \mathbf{X}\rangle \quad (n \geq 0). \tag{4}$$

In order to give a quantum-mechanical derivation of Hannay's angle, we look for quantum operators which are susceptible to describe the classical action I and angle θ variables. Since in the classical limit the action I is related to the excitation number n by $I = n\hbar$, it is natural to associate with it the quantum operator $I(\mathbf{X}) = \hbar N(\mathbf{X})$. The search for a quantum description of θ (which is canonically conjugate to $I : \{\theta, I\}_{\text{PB}} = 1$) is much harder. One can neither find a corresponding Hermitian operator $\Theta(\mathbf{X})$ such that $[\Theta(\mathbf{X}), N(\mathbf{X})] = i$ nor even a unitary operator $U(\mathbf{X})$ (in place of $\exp(-i\Theta(\mathbf{X}))$) such that $[U(\mathbf{X}), N(\mathbf{X})] = U(\mathbf{X})$. This difficulty, also encountered in the general problem of defining a phase operator in quantum physics, is connected with the fact that the spectrum of $N(\mathbf{X})$ is bounded from below. However, as in the case of the harmonic oscillator, an approximate solution can be found, which becomes exact in the classical limit. Let $U(\mathbf{X})$ be the isometric operator defined by [9]:

$$\begin{aligned} U(\mathbf{X})|n, \mathbf{X}\rangle &= |n - 1, \mathbf{X}\rangle \quad (n \geq 1) \\ U(\mathbf{X})|0, \mathbf{X}\rangle &= 0. \end{aligned} \tag{5}$$

It satisfies the relations:

$$\begin{aligned} [U(\mathbf{X}), N(\mathbf{X})] &= U(\mathbf{X}) \\ [U(\mathbf{X}), H(\mathbf{X})]|n + 1, \mathbf{X}\rangle &= (E_{n+1}(\mathbf{X}) - E_n(\mathbf{X}))U(\mathbf{X})|n + 1, \mathbf{X}\rangle. \end{aligned} \tag{6}$$

In the classical limit ($\hbar \rightarrow 0, n \rightarrow \infty, n\hbar = I$ finite) one can forget the second equality in (5) and consider $U(\mathbf{X})$ to be unitary. Moreover, since the distance between consecutive energy levels is of the order of \hbar , in this limit, the second equality in (6) reads $[U(\mathbf{X}), H(\mathbf{X})] = \hbar\omega(I, \mathbf{X})U(\mathbf{X})$ where $\omega(I, \mathbf{X}) = (\partial/\partial I)E_n(\mathbf{X})$; therefore it is the quantum analogue of the classical equation $\dot{\theta} = \{\theta, H\}_{\text{PB}} = \omega$ for the angle variable θ . In conclusion, $\hbar N(\mathbf{X})$ and $U(\mathbf{X})$ are the quantum operators which are naturally related to the classical variables I and $\exp(-i\theta)$.

Coherent states are also well known to be appropriate for the classical limit. They are usually defined for the harmonic oscillator, due to the particular role played in physics by the position and momentum variables q and p . However for any Hamiltonian $H(\mathbf{X})$, one can as well define associated coherent states by:

$$|\alpha, \mathbf{X}\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, \mathbf{X}\rangle \tag{7}$$

where α is an arbitrary complex number†. As $|\alpha|^2$ is nothing but the mean value of the operator $N(\mathbf{X})$ in these states, the classical limit now corresponds to \hbar goes to zero, $|\alpha|$ goes to infinity, the product $|\alpha|^2\hbar = I$ remaining finite. According to the Stirling approximation the sum over n in (7) is highly peaked around the value $N = |\alpha|^2$. The value $\Delta n = |\alpha|$ of the dispersion in n shows that a coherent state still ‘covers’, in the classical limit, an infinite number of eigenvectors, but that it corresponds to an infinitesimal fluctuation $\Delta I = [\hbar I]^{1/2}$ of the action. With the help of this remark, one easily verifies that $\exp(i \arg(\alpha))$ is nothing but the mean value of the operator $U(\mathbf{X})$ in the coherent state $|\alpha, \mathbf{X}\rangle$. Therefore, in the classical limit, the complex parameter α of this coherent state is related to the action-angle variables of the Hamiltonian by

$$\alpha = \sqrt{\frac{I}{\hbar}} \exp(-i\theta). \quad (8)$$

(In the next section, we give a supplementary justification of (8) based on the study of the evolution in time of the coherent states.) It will be important for the following sections to remember, that, in the classical limit, a coherent state $|\alpha, \mathbf{X}\rangle$ (or better a ray $|\overline{\alpha, \mathbf{X}}\rangle$) is in correspondence with a point in phase space. The set of rays $\{|\alpha \exp i\varphi, \mathbf{X}\rangle\}$ (α fixed, $0 \leq \varphi < 2\pi$) is associated with a curve $C(I, \mathbf{X})$ of constant action, and the particular ray $|\overline{\alpha, \mathbf{X}}\rangle$ corresponds to the origin for the angle variable on this curve. This correspondence of course depends on the choice of the family $\{|n, \mathbf{X}\rangle\}$. (This point is discussed in appendix 1.)

Finally let us remark that the above analysis can be trivially generalized to any integrable Hamiltonian in the same way as one extends the description of the one-dimensional harmonic oscillator to the case of p degrees of freedom (or to an infinite number in field theory [10]). The coherent states $|\alpha, \mathbf{X}\rangle$ ($\alpha = (\alpha_1, \dots, \alpha_p) \in \mathcal{C}^p$) are approximate eigenvectors of the operators $\hbar N_i(\mathbf{X})$ and $U_i(\mathbf{X})$ ($i = 1, \dots, p$) which describe the classical observables I_i and $\exp(-i\theta_i)$, (θ_i, I_i) being the i th pair of action-angle variables.

3. Hannay’s angle and coherent states

In this section, we give two derivations of the relation (3) between Berry’s phase and Hannay’s angle, using the coherent states $|\alpha, \mathbf{X}\rangle$. Whereas in the first one, the adiabatic evolution of the states $|n, \mathbf{X}(0)\rangle$ is taken for granted, in the second one, the adiabatic evolution of the coherent states $|\alpha, \mathbf{X}(0)\rangle$ is directly obtained from a variational principle.

Starting with the result of [1] describing the adiabatic evolution of the energy eigenstate $|n, \mathbf{X}(0)\rangle$:

$$\mathcal{U}(t)|n, \mathbf{X}(0)\rangle = \exp(i\Phi_n(t))|n, \mathbf{X}(t)\rangle \quad (9)$$

($\Phi_n(t)$ being given by relation (1) with $\Phi_n(0) = 0$), one deduces that a coherent state $|\alpha, \mathbf{X}(0)\rangle$ evolves according to:

$$\mathcal{U}(t)|\alpha, \mathbf{X}(0)\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_n \frac{\alpha^n \exp(i\Phi_n(t))}{\sqrt{n!}} |n, \mathbf{X}(t)\rangle. \quad (10)$$

† In (7) we have explicitly supposed that the number of eigenstates is strictly infinite. If this is not the case for a given value of \hbar , one knows that this number goes to infinity when \hbar goes to zero. So, in any case, formula (7) is meaningful in the classical limit. We acknowledge one of the referees for bringing our attention to this point.

As already mentioned, in the classical limit, the sum over n is highly peaked around the value $N = |\alpha|^2$. This allows $\Phi_n(t)$ to be replaced in this region by its first order approximation:

$$\Phi_n(t) = \Phi_N(t) + (n - N) \frac{\partial}{\partial N} \Phi_N(t) \tag{11}$$

and (10) becomes:

$$\mathcal{U}(t)|\alpha, \mathbf{X}(0)\rangle = \exp \left[i \left(\Phi_N(t) - N \frac{\partial}{\partial N} \Phi_N(t) \right) \right] |\alpha \exp i \frac{\partial}{\partial N} \Phi_N(t), \mathbf{X}(t)\rangle. \tag{12}$$

Therefore, up to a global phase factor, the evolved state of a coherent state associated with the initial Hamiltonian $\mathbf{H}(\mathbf{X}(0))$ is a coherent state $|\alpha(t), \mathbf{X}(t)\rangle$ associated with the Hamiltonian $\mathbf{H}(\mathbf{X}(t))$ at time t . Setting $\alpha(t) = |\alpha(t)| \exp -i\theta(t)$, one verifies that the modulus $|\alpha(t)|$ does not depend on time and one recovers the adiabatic invariance of the action I . As for the angular variable $\theta(t) = \theta(0) - (\partial/\partial N)\Phi_N(t)$, it evolves in time according to the relation

$$\dot{\theta}(t) = \hbar^{-1} \frac{\partial}{\partial N} E_N(\mathbf{X}(t)) - \frac{\partial}{\partial N} \dot{\gamma}_N^B(t). \tag{13}$$

If the Hamiltonian does not depend on time, (13) reduces to the classical Hamiltonian equation $\dot{\theta} = \omega(I, \mathbf{X})$; this is an additional justification of the relation (8) of the previous section. In the general case, the angular velocity $\dot{\theta}$ appears as the sum of the 'dynamical' part $\omega(I, \mathbf{X})$ and of an additional angular velocity

$$\dot{\theta}_I^H(t) = -\hbar \frac{\partial}{\partial I} \dot{\gamma}_N^B(t) \tag{14}$$

which we may call Hannay's angular velocity. When integrated over time, relation (14) leads to (3). This derivation of the relation between Berry's phase and Hannay's angle, which can be trivially extended to any integrable system through the use of the multidimensional coherent states $|\alpha, \mathbf{X}\rangle$, is up to our knowledge the simplest which has been given.

It is also interesting to give an autonomous derivation of (13), which does not presuppose the evolution of the energy eigenstates. The idea is to use the variational principle

$$\delta \left[\int \langle \Psi(t) | \left(i\hbar \frac{\partial}{\partial t} - \mathbf{H} \right) | \Psi(t) \rangle dt \right] = 0 \tag{15}$$

with a judicious selection of the test vectors $|\Psi(t)\rangle$ [11]. As is well known, this principle leads to Schrödinger's equation if there is no restriction on $|\Psi(t)\rangle$. However, if one pictures classical mechanics as some blurred version of quantum mechanics, it is natural to restrict the choice of $|\Psi(t)\rangle$ to 'classical states'. The coherent states are good candidates since they are in correspondence with the points of the classical phase space. We choose the coherent states $|\alpha(t), \mathbf{X}(t)\rangle$ because we want to parametrize the points in phase space, at time t , by the action and angle variables corresponding to

the Hamiltonian $\mathbf{H}(\mathbf{X}(t))$ at that time. The variation is with respect to the complex function $\alpha(t)$. A straightforward calculation shows that the bracket in (15) reads:

$$\begin{aligned} \left\langle \Psi(t) \left| \left(i\hbar \frac{\partial}{\partial t} - \mathbf{H} \right) \Psi(t) \right\rangle &= \frac{i\hbar}{2} [\bar{\alpha}\dot{\alpha} - \alpha\dot{\bar{\alpha}}] - \exp(-|\alpha|^2) \sum_n \frac{|\alpha|^{2n}}{n!} E_n(\mathbf{X}(t)) \\ &+ i\hbar \exp(-|\alpha|^2) \sum_{n,m} \frac{\bar{\alpha}^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \left\langle m, \mathbf{X}(t) \left| \frac{\partial}{\partial t} \right| n, \mathbf{X}(t) \right\rangle. \end{aligned} \tag{16}$$

When the parameters \mathbf{X} are independent of time the functional derivative of the right-hand side of (16) with respect to $\bar{\alpha}$ is

$$i\hbar\dot{\alpha} + \alpha \exp(-|\alpha|^2) \sum_n (n!)^{-1} |\alpha|^{2n} (E_n - E_{n+1})$$

and the variational principle, in the classical limit, leads to:

$$\dot{\alpha}(t) = -i\hbar^{-1} \frac{\partial}{\partial N} E_N(\mathbf{X}) \alpha(t) \quad (|\alpha|^2 = N). \tag{17}$$

It implies that the action $I = \hbar|\alpha|^2$ is invariant and that the angular velocity $\dot{\theta}$ is nothing but the classical one $\omega(I, \mathbf{X})$. When the parameters depend on time, but much more slowly than the angle $\theta(t)$ (adiabatic hypothesis), one can neglect in (16) the terms $m \neq n$ which appear in the double sum over m and n (this is analogous to the random phase approximation). If one introduces the explicit expression of Berry's frequency [1]:

$$\dot{\gamma}_n^B(t) = i \left\langle n, \mathbf{X}(t) \left| \frac{\partial}{\partial t} \right| n, \mathbf{X}(t) \right\rangle \tag{18}$$

this sum then reads $\hbar \exp -|\alpha|^2 \sum_n (n!)^{-1} |\alpha|^{2n} \dot{\gamma}_n^B$. So, taking into account the adiabatic evolution of the parameters amounts to replace E_n by $E_n - \hbar \dot{\gamma}_n^B$. Formula (17) becomes

$$\dot{\alpha}(t) = -i\hbar^{-1} \frac{\partial}{\partial N} (E_N(\mathbf{X}(t)) - \hbar \dot{\gamma}_N^B(t)) \alpha(t). \tag{19}$$

One recovers the property that the action $I = \hbar|\alpha|^2$ is invariant and the angular velocity $\dot{\theta}$ the sum of a dynamical part $\omega(I, \mathbf{X})$ and of Hannay's angular velocity (14).

4. Hannay's angle and geometrical transport

In the previous section Berry's phase and Hannay's angle have been considered in the dynamical context of adiabatic evolution. However there also exists a pure geometrical approach which was first given for Berry's phase [2]. We recall shortly how this phase is associated with the transport of the eigenstates $|n, \mathbf{X}\rangle$ because this will be useful to understand how Hannay's angle is associated with the transport of classical trajectories in our coherent state formalism.

In a Hilbert space, equipped with the usual distance deduced from the Hermitian product, let us consider two infinitesimally close normalized vectors $|\Psi_0\rangle$ and $|\Psi_0 +$

$d\Psi_0\rangle$, chosen as reference states on the two corresponding rays $\overline{|\Psi_0\rangle}$ and $\overline{|\Psi_0 + d\Psi_0\rangle}$. There is a natural way to associate with any given vector $|\Psi\rangle = |\Psi_0\rangle \exp(i\varphi_0)$ of the ray $\overline{|\Psi_0\rangle}$ one vector $|\Psi + d\Psi\rangle = |\Psi_0 + d\Psi_0\rangle \exp[i(\varphi_0 + d\varphi_0)]$ of the ray $\overline{|\Psi_0 + d\Psi_0\rangle}$. By definition it is the one among all vectors $|\Psi_0 + d\Psi_0\rangle \exp i(\varphi_0 + d\varphi)$ which is closest to $|\Psi_0\rangle \exp i\varphi_0$. This vector is obtained by minimizing the quadratic quantity $|||d\Psi_0\rangle + i|\Psi_0\rangle d\varphi||^2$ with respect to $d\varphi$. The result:

$$d\varphi_0 = i\langle\Psi_0 | d\Psi_0\rangle \tag{20}$$

which can be equivalently written

$$\langle\Psi | d\Psi\rangle = 0 \tag{21}$$

defines the transport of the vector $|\Psi\rangle$ onto the vector $|\Psi + d\Psi\rangle$ and more generally the transport of any vector of the ray $\overline{|\Psi_0\rangle}$ onto the corresponding vector of the ray $\overline{|\Psi_0 + d\Psi_0\rangle}$. The transport equation (21) may also be deduced from the notion of distance between infinitesimally close rays, a method that we shall use further. Indeed, let us define the squared distance d [12]:

$$d^2(\overline{|\Psi_0\rangle}, \overline{|\Psi_0 + d\Psi_0\rangle}) = \langle d\Phi | d\Phi\rangle - |\langle\Phi | d\Phi\rangle|^2 \tag{22}$$

where $|\Phi\rangle$ and $|\Phi + d\Phi\rangle$ are any infinitesimally close vectors belonging to the rays $\overline{|\Psi_0\rangle}$ and $\overline{|\Psi_0 + d\Psi_0\rangle}$. (One can verify that the right-hand side of (22) is independent of the choice of the pair $|\Phi\rangle, |\Phi + d\Phi\rangle$.) It is then obvious that $\langle d\Phi | d\Phi\rangle$ is at a minimum when the transport equation $\langle\Phi | d\Phi\rangle = 0$ is satisfied. If instead of two close vectors $|\Psi_0\rangle$ and $|\Psi_0 + d\Psi_0\rangle$ one now considers a family $\{|\Psi_0(\mathbf{X})\rangle\}$ of quantum states which depend smoothly on \mathbf{X} and are chosen as reference states for the family $\{\overline{|\Psi_0(\mathbf{X})\rangle}\}$ of rays, the relation (20) allows the definition of the transport of a vector of the ray $\overline{|\Psi_0(\mathbf{X}(0))\rangle}$ along a curve $\mathbf{X}(t)$. The transported vector $|\Psi(\mathbf{X}(t))\rangle = |\Psi_0(\mathbf{X}(t))\rangle \exp(i\varphi_0(t))$ of the initial vector $|\Psi_0(\mathbf{X}(0))\rangle \exp(i\varphi_0(0))$ has a phase $\varphi_0(t)$ which satisfies the differential equation:

$$\frac{\partial}{\partial t} \varphi_0(t) = i \left\langle \Psi_0(\mathbf{X}(t)) \left| \frac{\partial}{\partial t} \right| \Psi_0(\mathbf{X}(t)) \right\rangle. \tag{23}$$

The right-hand side of (23) is nothing other than Berry's frequency (18) when one takes for $|\Psi_0(\mathbf{X})\rangle$ the energy eigenvectors $|n, \mathbf{X}\rangle$. (The same result could as well be obtained from the relation $\langle\Psi(\mathbf{X}(t)) | (\partial/\partial t) | \Psi(\mathbf{X}(t))\rangle = 0$.)

We now come to the geometrical transport of classical tori. We characterize each trajectory in the phase space by three data: the parameters \mathbf{X} , the action I and an origin on the curve for the angular variable. At this stage of the discussion, it is important to recall that once the eigenstates $|n, \mathbf{X}\rangle$ are chosen, each coherent ray $|\alpha, \mathbf{X}\rangle$ corresponds to one definite point in phase space. We call $C(\theta, I, \mathbf{X})$ that trajectory which is deduced from the reference trajectory $C(I, \mathbf{X})$ by an angular shift θ . (Trajectories with the same value of \mathbf{X} and I differ from each other by the value of θ in the same way as vectors of the ray $|n, \mathbf{X}\rangle$ differ from each other by a global phase factor.) The transport of a classical trajectory can be determined by the transport of any one of its points. Let I and θ_0 be the action-angle variables of such a point and $I + dI, \theta_0 + d\theta$ those of the transported point (θ_0 and $\theta_0 + d\theta$ are measured respectively

on the reference curves $C(I, \mathbf{X})$ and $C(I + dI, \mathbf{X} + d\mathbf{X})$. The condition for this transport of points to define a transport of trajectories is that dI and $d\theta$ do not depend on θ_0 . The corresponding infinitesimal transport at the level of the coherent rays is such that the ray $|\alpha, \mathbf{X}\rangle (\hbar|\alpha|^2 = I; \arg(\alpha) = -\theta_0)$ becomes $|\alpha + d\alpha, \mathbf{X} + d\mathbf{X}\rangle (\hbar|\alpha + d\alpha|^2 = I + dI, \arg(\alpha + d\alpha) = -(\theta_0 + d\theta))$. In order to define the transport of a trajectory $C(\theta_0, I, \mathbf{X})$, which corresponds to the set of rays $\{|\alpha \exp(i\varphi), \mathbf{X}\rangle\}$ ($\varphi \in [0, 2\pi]$), onto the trajectory $C(\theta_0 + d\theta, I + dI, \mathbf{X} + d\mathbf{X})$, which corresponds to the set $\{|\alpha + d\alpha \exp(i\varphi), \mathbf{X} + d\mathbf{X}\rangle\}$, we introduce a distance between these two sets of rays. The square distance dI^2 between them is the average over the trajectory (i.e. over φ) of the square distances between a ray in the first set and its transported one in the second set:

$$dI^2 = \frac{1}{2\pi} \int_0^{2\pi} d^2(\overline{|\alpha e^{i\varphi}, \mathbf{X}\rangle}, \overline{|\alpha + d\alpha e^{i\varphi}, \mathbf{X} + d\mathbf{X}\rangle}) d\varphi. \quad (24)$$

The transport equation is obtained by the minimization of dI^2 with respect to dI and $d\theta$.

The distance dI^2 is a quadratic function of $d|\alpha|$, $d\theta$ and $d\mathbf{X}$. If $d\mathbf{X} = 0$, the distance between two coherent rays $|\alpha, \mathbf{X}\rangle$ and $|\alpha + d\alpha, \mathbf{X}\rangle$ is the same as that between the usual coherent rays $|\alpha\rangle$ and $|\alpha + d\alpha\rangle$ of the harmonic oscillator, since one can deduce the former from the latter by a unitary transformation (the one which brings the eigenvectors $|n\rangle$ of the harmonic oscillator onto those of $\mathbf{H}(X), |n, \mathbf{X}\rangle$). This distance is simply: $(d|\alpha|)^2 + |\alpha|^2 d\theta^2$ [12]. More generally, when $d\mathbf{X}$ is different from zero, the distance dI^2 reads (cf appendix 2)

$$dI^2 = (d|\alpha|)^2 + |\alpha|^2 d\theta^2 - 2\mathbf{A} \cdot d\mathbf{X} d|\alpha| - 2\mathbf{B} \cdot d\mathbf{X} d\theta + C_{ij} dX_i dX_j \quad (25)$$

where

$$\begin{aligned} \mathbf{A} &= 0 \\ \mathbf{B} &= i|\alpha|^2 \frac{\partial}{\partial N} \left\langle N, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| N, \mathbf{X} \right\rangle \quad (\text{with } |\alpha|^2 = N). \end{aligned} \quad (26)$$

The result of the minimization is then

$$dI = 0 \quad d\theta = i \frac{\partial}{\partial N} \langle N, \mathbf{X} | d | N, \mathbf{X} \rangle. \quad (27)$$

Therefore, when the transport is realized along a curve Γ in the space of parameters, one recovers the expected relations:

$$\dot{I} = 0 \quad \dot{\theta}(t) = \dot{\theta}_I^H(t). \quad (28)$$

This clearly shows that the transport of classical trajectories can be derived from the minimization of a distance and is entirely described by Hannay's angle (the action I being invariant in the transport) in exactly the same way as the transport of energy eigenvectors is entirely described by Berry's phase (the excitation number n being kept fixed).

Let us finally show that the relation $\langle \Psi, d\Psi \rangle = 0$ which also defines the quantum transport, has, as well, an analogue at the level of coherent states. The easiest way to

find it is to come back to the variational principle (15) and to eliminate any reference to dynamics; i.e. we set $\mathbf{H} = 0$ in it. Then providing one may ignore in (16) the terms $m \neq n$, the calculation which lead to (19), trivially gives the transport relation (28). In section 3, this omission was justified by the adiabatic hypothesis. In order to make it rigorous, it suffices now to replace the variational principle (15) by its averaged version:

$$\delta \left[\int \left\langle \alpha(t)e^{i\varphi}, \mathbf{X}(t) \left| i \frac{\partial}{\partial t} \right| \alpha(t)e^{i\varphi}, \mathbf{X}(t) \right\rangle d\varphi \right] = 0. \tag{29}$$

This replacement is justified since we now want to transport a trajectory, i.e. a whole set of coherent rays $\{|\overline{\alpha \exp(i\varphi)}, \mathbf{X}(t)\rangle\}, (\varphi \in [0, 2\pi])$. One can easily verify that, effectively, the variational principle (29) does not depend on the representation which one chooses for these rays. Let us remark that if instead of a variational principle one imposes that the integrant $\langle \Psi | d\Psi \rangle$ in (29) is zero one obtains the relation $|\alpha|^2 \dot{\theta} = \dot{\gamma}_n^B$ describing a 'surplus area' [7]. Although this relation leads to (28) when differentiated with respect to I , it is in fact incompatible with it except when $\dot{\gamma}_n^B$ is proportionnal to n . The necessity of defining the classical transport by (29) and not by applying $\langle \Psi, d\Psi \rangle = 0$ to the classical coherent states is quite understandable if one notes that the latter depends on the representations which are chosen on the coherent rays.

5. Conclusion

In this paper, we have developed a framework which allows a parallel description of Berry's phase and Hannay's angle. Thanks to the coherent state formalism, we have been able to transpose at the quantum level the problem of the geometrical transport of classical tori. One remarkable fact is that this transport can be obtained from the minimization of a distance. It would be interesting to define such a distance in the phase space from purely classical arguments. In particular, one may expect that its minimization leads to the connection which has been defined by Montgomery [5].

Appendix 1

In this appendix, we study the consequences of the choice of the family $|n, \mathbf{X}\rangle$ of eigenvectors of the Hamiltonians $H(\mathbf{X})$ on the correspondence between the coherent rays $|\overline{\alpha}, \overline{\mathbf{X}}\rangle$ and the points in phase space. Let us consider the gauge transformation

$$|n, \mathbf{X}'\rangle = \exp(i\varphi_n(\mathbf{X}))|n, \mathbf{X}\rangle \tag{A1.1}$$

defining the new family $\{|n, \mathbf{X}'\rangle\}$, and let $|\alpha, \mathbf{X}'\rangle$ be the new coherent states

$$|\alpha, \mathbf{X}'\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, \mathbf{X}'\rangle. \tag{A1.2}$$

In the classical limit, the sum in (A1.2) is obtained from a first order expansion of $\varphi_n(\mathbf{X})$ around the value $N = |\alpha|^2$ in the same way as formula (10) was calculated in section 3. One gets the relation between coherent rays (analogous to (12)):

$$|\overline{\alpha}, \overline{\mathbf{X}}'\rangle = \overline{|\alpha \exp i(\partial/\partial N)\varphi_N(\mathbf{X}), \mathbf{X}'\rangle}. \tag{A1.3}$$

Therefore the ray $\overline{|\alpha, \mathbf{X}\rangle}$ corresponds to a point in phase space which still lies on the curve $C(I, \mathbf{X})$ but whose angular coordinate has been shifted by $\theta(I, \mathbf{X}) = -(\partial/\partial N)\varphi_N(\mathbf{X})$. In other words the change $\{|n, \mathbf{X}\rangle\} \rightarrow \{|n, \mathbf{X}\rangle'\}$ of the family of eigenstates corresponds to a change $C(I, \mathbf{X}) \rightarrow C(\theta, I, \mathbf{X})$ (i.e. a 'rotation' with respect to the angle variable) of the reference trajectories. Of course, although Hannay's angle, defined by (2), depends on this choice, neither its value for a closed curve Γ in the space of parameters, nor the geometrical transport of trajectories depend on it.

Appendix 2

We calculate those terms in the distance (25) which allow us to obtain the classical transport (28). Let $|\Phi\rangle = |\alpha, \mathbf{X}\rangle$ ($\alpha = |\alpha| \exp -i\theta$) and $|\Phi + d\Phi\rangle = |(\alpha + d\alpha) \exp -i(\theta + d\theta)\rangle$ be two close coherent states. Then dI^2 reads:

$$dI^2 = \overline{\langle d\Phi | d\Phi \rangle} - |\overline{\langle \Phi | d\Phi \rangle}|^2 \tag{A2.1}$$

where the bar means an average of the metric elements over the argument of the complex number α . From expression (7) of $|\Phi\rangle$, one easily deduces

$$d\Phi = \exp(-|\alpha|^2/2) \sum_n \frac{\alpha^n}{\sqrt{n!}} \left[\left(\frac{n}{|\alpha|} - |\alpha| \right) d|\alpha| + in d\theta + d\mathbf{X} \cdot \frac{\partial}{\partial \mathbf{X}} \right] |n, \mathbf{X}\rangle \tag{A2.2}$$

and

$$\langle \Phi, d\Phi \rangle = i|\alpha|^2 d\theta + \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n,m} \frac{\bar{\alpha}^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \left\langle m, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| n, \mathbf{X} \right\rangle d\mathbf{X}. \tag{A2.3}$$

The term $2\mathbf{A} \cdot d\mathbf{X} d|\alpha|$ in (25) originates from $\overline{\langle d\Phi | d\Phi \rangle}$. Taking the average into account, one sees that this term is the sum over n of quantities proportional to: $(\langle n, \mathbf{X} | (d/d\mathbf{X}) | n, \mathbf{X} \rangle + CC)$. Since the vectors are normalized, all these sums are zero. Therefore

$$\mathbf{A} = 0. \tag{A2.4}$$

The term $2\mathbf{B} \cdot d\mathbf{X} d\theta$ in (25) originates from $\overline{\langle d\Phi | d\Phi \rangle}$ and $|\overline{\langle \Phi | d\Phi \rangle}|^2$. Their sum is

$$\begin{aligned} 2\mathbf{B} &= \exp(-|\alpha|^2) \sum_n \frac{|\alpha|^{2n}}{n!} (-in + i|\alpha|^2) \left\langle n, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| n, \mathbf{X} \right\rangle + CC \\ &= 2|\alpha|^2 \exp(-|\alpha|^2) \sum_n \frac{|\alpha|^{2n}}{n!} \left[\left\langle n, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| n, \mathbf{X} \right\rangle \right. \\ &\quad \left. - \left\langle n+1, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| n+1, \mathbf{X} \right\rangle \right]. \end{aligned} \tag{A2.5}$$

Since, in the classical limit, the sum in (A2.5) is peaked around the value $N = |\alpha|^2$, one gets:

$$\mathbf{B} = -|\alpha|^2 \frac{\partial}{\partial N} \left\langle N, \mathbf{X} \left| \frac{\partial}{\partial \mathbf{X}} \right| N, \mathbf{X} \right\rangle. \tag{A2.6}$$

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